Influence of Initial Stress on the Vibrations of Simply Supported Circular Cylindrical Shells

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The influence of initial uniform circumferential and axial stress on the frequency of vibrations of simply supported circular cylindrical shells has been previously investigated on the basis of shell theories, which can be anticipated to yield accurate results only for modes having a large number of circumferential waves. In this investigation, a bending theory presented by Herrmann and Armenàkas is employed in re-examining the influence of initial uniform circumferential and axial stress on the frequency of vibrations of simply supported circular cylindrical shells. The results are compared with those of previous investigations.

Nomenclature

x, θ, z = initial coordinates of a point
R = radius of the middle surface of the shell
h = shell thickness
L = axial length of the shell
u, v, w = displacement components of the middle surface in the axial, circumferential, and radial directions, respectively
ΔF_x, ΔF_θ, Δq = axial, circumferential, and radial components, respectively, of the change of the initial shell surface tractions, due to deformation, taken per unit undeformed middle surface area
m_x, m_θ = shell moment due to the axial and circumferential traction, respectively
Δm_x, Δm_θ = changes in m_x and m_θ, respectively, due to deformation
N = initial uniform circumferential normal resultant-stress
$N_{xx}{}^a M_{xx}{}^a$ = additional axial resultant-stress and resultant-moment, respectively
T = initial uniform axial normal resultant-stress
p = uniform lateral pressure
ρ = mass density
t = time
ω = frequency of vibration
ω_s = frequency of the first simple thickness-shear mode of an infinite plate of thickness h
Ω = ω/ω_s = frequency ratio
E_p = $[Eh/(1 - \nu^2)]D = Eh^3/[12(1 - \nu^2)]$ = shell extensional and flexural modulus, respectively
E, G = Young's and shear modulus, respectively
ν = Poisson's ratio
n = number of circumferential waves
m = number of axial half-waves
β = $\pi Rm/Ln$ = ratio of the circumferential to the axial wavelength
s = $h/2R$

Introduction

IN the last decade, the increase in the use of cylindrical shells in structural components of aircraft, missiles, submarines, and rockets has stimulated many investigators to reconsider the dynamic behavior of initially unstressed cylindrical shells. However, the related problem of shells vibrating under the influence of initial stresses has not received, as yet, the attention it merits.

Linear flexure theories, encompassing the effect of initial, constant, axial, and circumferential (due to hydrostatic pressure) shell stresses were established by Timoshenko[1] and Flügge[2] by considering the equilibrium of a deformed shell element. In deriving the Timoshenko displacement equations of motion,[1] the stress-strain relations of Love's first approximation are used, resulting in nonsymmetric equations and, thus, precluding the existence of a strain potential. Because of geometric complexities, however, the theories of Timoshenko and Flügge differ in terms involving initial stress whose order of magnitude is $1/n^2$ and h^2/R^2. Donnell[3] disregarded all these higher-order terms and derived an uncoupled eight-order differential equation of equilibrium involving only the radial displacement component.

Recently, on the basis of the nonlinear three-dimensional theory of elasticity, Herrmann and Armenàkas[4] presented several linear theories of motion of elastic cylindrical shells subjected to a general state of initial stress. These theories are more inclusive than most others heretofore available inasmuch as they consider the initial stresses to be functions of the space coordinates, and, in addition to the effect of initial membrane stresses, they encompass that of initial moments and transverse shear.

For the case of a shell under initial hydrostatic pressure and uniform axial stress, the bending theory presented in Ref. 4 reduces to the recently modified Flügge theory.[5] The theory in Ref. 4 has been applied by Armenàkas and Herrmann[6] to study the effect of initial uniform circumferential stress, bending moment, and radial shear on the dynamic response of an infinitely long elastic shell.

The effect of initial circumferential and longitudinal stresses on the frequency of vibrations of simply supported circular cylindrical shells has been studied by Reissner.[7,8] In Ref. 7 he obtained a simple expression for the frequency of the radial mode on the basis of the Marguerre shallow shell theory by omitting the effect of transverse and longitudinal inertia. In Ref. 8 Reissner established the frequency equation for shells under all around pressure (axial stress equals one-half circumferential) by using a shallow shell membrane theory wherein he retained the effect of longitudinal and transverse inertia. However, since Reissner's results in both cases are based upon shallow shell theories, they are not expected to be valid for modes having a small number of circumferential waves.

Fung, Sechler, and Kaplan[9] reinvestigated the effect of all around pressure on the frequency of vibrations of simply supported circular cylindrical shells. However, the frequency equation was established on the basis of the Timoshenko

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theory[1] wherein it can be shown that certain terms whose relative order of magnitude is $1/n^2$ are incorrect. For certain ranges of the shell parameters, therefore, for modes having a small number of circumferential waves, the results of Fung, Sechler, and Kaplan cannot be anticipated to be more accurate than those obtained by Reissner.

Thus, it is evident that a systematic re-examination of the vibrations of simply supported circular cylindrical shells under the influence of initial stresses is appropriate. Hence, the equations derived in Ref. 4 are employed in studying the effect of uniform axial and circumferential stress on the frequency of vibrations of simply supported shells. Furthermore, the effect on the motion of the change due to deformation of the magnitude and direction of the applied load is illustrated and discussed by considering the uniform lateral pressure that induces the circumferential stress as 1) hydrostatic, the direction and magnitude per unit original area alter due to deformation and 2) constant directional, the direction and magnitude per unit original area remain unchanged during deformation.

Equations of Motion

A circular cylindrical shell of length L, constant thickness h, and mean radius R is referred to a system of modified cylindrical coordinates x, θ, z. x is measured along the axis of the shell, θ along the circumference, and z radially outward from the middle surface.

The general deformed configuration of the shell (referred to as the final state of stress) is attained from the unstressed and unstrained state by passing through an intermediate equilibrium state, the state of initial stress. The x, θ, and z components of displacement, from the initial to the final state of stress, of a point on the middle surface of the shell, are designated by u, v, and w, respectively. The displacement equations of motion that include the effect of initial uniform circumferential normal resultant-stress N and initial uniform axial normal resultant-stress T may be obtained from Eqs. (32) of Ref. 4, and they are

$$\left.\begin{aligned}
&\left[E_p \frac{\partial^2}{\partial x^2} + \frac{Gh}{R^2}\left(1+\frac{h^2}{12R^2}\right)\frac{\partial^2}{\partial\theta^2} - \rho h \frac{\partial^2}{\partial t^2} + T\frac{\partial^2}{\partial x^2} + \frac{N}{R^2}\frac{\partial^2}{\partial\theta^2}\right]u + \frac{1+\nu}{2R}E_p\frac{\partial^2 v}{\partial x\partial\theta} + \left[\nu\frac{E_p}{R} - \frac{D}{R}\frac{\partial^2}{\partial x^2} + \frac{Gh^2}{12R^3}\frac{\partial^2}{\partial\theta^2}\right]\frac{\partial w}{\partial x} + \Delta F_x = 0\\
&E_p\left(\frac{1+\nu}{2R}\right)\frac{\partial^2 u}{\partial x\partial\theta} + \left[\frac{E_p}{R^2}\frac{\partial^2}{\partial\theta^2} + Gh\left(1+\frac{h^2}{4R^2}\right)\frac{\partial^2}{\partial x^2} - \rho h\frac{\partial^2}{\partial t^2} + T\frac{\partial^2}{\partial x^2} + \frac{N}{R^2}\frac{\partial^2}{\partial\theta^2} - \frac{N}{R^2} - \frac{m_z}{R^2}\right]v + \left[\frac{E_p}{R^2} - \frac{D(3-\nu)}{2R^2}\frac{\partial^2}{\partial x^2} + \frac{2N}{R^2} + \frac{m_z}{R^2}\right]\frac{\partial w}{\partial\theta} + \Delta F_\theta + \frac{\Delta m_\theta}{R} = 0\\
&\left[\frac{\nu E_p}{R} - \frac{D}{R}\frac{\partial^2}{\partial x^2} + \frac{Gh^2}{12R^2}\frac{\partial^2}{\partial\theta^2}\right]\frac{\partial u}{\partial x} + \left[E_p - D\left(\frac{3-\nu}{2}\right)\frac{\partial^2}{\partial x^2} + 2N + m_z\right]\frac{1}{R^2}\frac{\partial v}{\partial\theta} + \left[\frac{E_p}{R^2} + \frac{D}{R^4} + D\frac{\partial^4}{\partial x^4} + \frac{2D}{R^2}\frac{\partial^4}{\partial x^2\partial\theta^2} + \frac{D}{R^4}\frac{\partial^4}{\partial\theta^4} + \frac{2D}{R^4}\frac{\partial^2}{\partial\theta^2} + \rho h\frac{\partial^2}{\partial t^2} - T\frac{\partial^2}{\partial x^2} + \frac{N}{R^2} - \frac{N}{R^2}\frac{\partial^2}{\partial\theta^2} - \frac{m_z}{R^2}\frac{\partial^2}{\partial\theta^2} - m_z\frac{\partial^2}{\partial x^2}\right]w - \frac{\partial\Delta m_x}{\partial x} - \frac{1}{R}\frac{\partial\Delta m_\theta}{\partial\theta} - \Delta q = 0
\end{aligned}\right\} \quad (1)$$

where ΔF_x, ΔF_θ, Δq are the axial, circumferential, and radial components, respectively, of the change of the initial shell surface tractions, due to deformation, expressed per unit undeformed middle surface area; Δm_x, Δm_θ are the axial and circumferential components, respectively, of the change, due to deformation, of the moment induced by the surface traction, expressed per unit undeformed middle surface area; m_z is the sum of the products of the radial component of the initial surface traction and the z coordinate, evaluated at the two surfaces of the shell, and expressed per unit undeformed middle surface area.

The equations of motion in Ref. 4 have been derived on the basis of the nonlinear theory of elasticity. The dependence of the additional displacement components on the thickness coordinate has been assumed linear. Linearization has been attained by disregarding the effect of the displacements associated with the initial stresses and by assuming that the unit elongations, shears, and rotations associated with the additional displacements are small as compared with unity; therefore, the effect of terms encompassing products of displacement gradients and additional stresses has been disregarded as compared to the additional stresses. The initial stresses, however, may be of a higher order of magnitude than the additional stresses, and, concomitantly, terms involving products of initial stresses and displacement gradients have been retained in comparison to the additional stresses.

Frequency Equations

For a shell under the influence of initial uniform lateral pressure, disregarding the bending effect, the condition of initial equilibrium yields

$$N = \pm pR[1 \mp (h/2R)] \quad (2)$$

The upper sign applies to internal pressure, whereas the lower sign applies to external pressure. Moreover, by definition, it is evident that

$$m_z = -(ph/2)[1 \mp (h/2R)] \quad (3)$$

For a shell under hydrostatic pressure, as shown in Ref. 10,

$$\left.\begin{aligned}
\Delta q &= \frac{N}{R}\left[\frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{\partial u}{\partial x} + \frac{w}{R} \pm \frac{h}{2}\left(\frac{w}{R^2} + \frac{1}{R^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial^2 w}{\partial x^2}\right)\right]\\
\Delta F_x &= -\frac{N}{R}\frac{\partial w}{\partial x}\\
\Delta m_x &= \pm\frac{h}{2}\frac{N}{R}\frac{\partial w}{\partial x}\\
\Delta F_\theta &= \frac{N}{R^2}\left(v - \frac{\partial w}{\partial\theta}\right)\\
\Delta m_\theta &= \mp\frac{h}{2}\frac{N}{R^2}\left(v - \frac{\partial w}{\partial\theta}\right)
\end{aligned}\right\} \quad (4)$$

In the case of constant directional pressure,

$$\Delta F_x = \Delta F_\theta = \Delta m_\theta = \Delta m_x = \Delta q = 0 \quad (5)$$

Solutions are assumed in the form

$$\begin{aligned}
u &= U\cos\left(\frac{m\pi x}{L}\right)\cos(n\theta)e^{i\omega t}\\
v &= V\sin\left(\frac{m\pi x}{L}\right)\sin(n\theta)e^{i\omega t}\\
w &= W\sin\left(\frac{m\pi x}{L}\right)\cos(n\theta)e^{i\omega t}
\end{aligned} \quad (6)$$

These displacements satisfy the conditions $v = w = N_{xx}{}^a =$

$M_{xx}^a = 0$ at the boundaries $x = 0$ and $x = L$. These conditions may be effectuated by thin, deep edge rings, rigid in their place of curvature and nonresistant to forces and moments normal to their own plane.

Introduction of (6) into the equilibrium equations (1), as modified in accordance with (4) in the case of hydrostatic pressure and (5) in the case of constant directional pressure, results in three homogeneous algebraic equations for the amplitude factors U, V, W. For a nontrivial solution, the determinant of their coefficients must vanish, resulting in the following frequency equation:

$$\Omega^6 + K_1\Omega^4 + K_2\Omega^2 + K_3 + K_4\bar{N} + K_5\bar{T} = 0 \qquad (7)$$

where

$$\Omega = \frac{\omega}{\omega_s} \qquad \omega_s = \frac{\pi}{h}\left(\frac{G}{\rho}\right)^{1/2} \qquad \bar{N} = \frac{N}{E_p} \qquad \bar{T} = \frac{T}{E_p} \qquad (8)$$

The reference frequency ω_s is the frequency of the first simple thickness-shear mode of an infinite plate of thickness h. Disregarding terms of the order of magnitude of $(N/E_p)^2$ and $(h/R)^2$ as compared to unity, the coefficients of the frequency equations are

$$\left.\begin{aligned}
K_1 &= \frac{-2}{(1-\nu)\pi^2}\left(\frac{h}{R}\right)^2\left[n^2\left(\frac{3-\nu}{2}\right)(1+\beta^2) + 1 + n^4\frac{s^2}{3}(1+\beta^2)^2\right]\\
K_2 &= \frac{2n^2}{\pi^4(1-\nu)}\left(\frac{h}{R}\right)^4\left[n^2(1+\beta^2)^2 + (3+2\nu)\beta^2 + 1 + \frac{n^4s^2}{3}\left(\frac{3-\nu}{1-\nu}\right)(1+\beta^2)^3\right]\\
K_3 &= \frac{-4n^4}{\pi^6(1-\nu)^2}\left(\frac{h}{R}\right)^6\left\{(1-\nu^2)\beta^4 + \frac{s^2}{3}\{(1+\beta^2)^4n^4 + 2(2-\nu)\beta^2 + 1 - 2n^2[\nu\beta^6 + 3\beta^4 + (4-\nu)\beta^2 + 1]\}\right\}\\
K_{4h} &= \frac{-4n^2}{\pi^6(1-\nu)^2}\left(\frac{h}{R}\right)^6\{n^4(1+\beta^2)^2 - n^2(1+3\beta^2) \pm sn^2[n^2(1+\beta^2)^3 - 1 - 2\beta^2]\}\\
K_{4c} &= \frac{-4n^2}{\pi^6(1-\nu)^2}\left(\frac{h}{R}\right)^6\left[n^4(1+\beta^2)^2 + 1 - 2n^2 + \beta^2[2+2\nu - n^2(3+2\nu)] \mp s\{n^4(1+\beta^2)^3 + 1 - 2n^2 + \beta^2[2+2\nu-2n^2(2+\nu)]\}\right]\\
K_5 &= \frac{-4n^4\beta^2}{\pi^6(1-\nu)^2}\left(\frac{h}{R}\right)^6[n^2(1+\beta^2)^2 + 1 + \beta^2(3+2\nu)]\\
s &= \frac{h}{2R} \qquad \beta = \frac{m\pi R}{Ln}
\end{aligned}\right\} \qquad (9)$$

Terms involving s^2 in the foregoing coefficients represent the bending effect; their omission results in a membrane theory that is not expected to yield acceptable results in cases where the motion is not predominantly extensional. Terms involving s account for the fact that the pressure is not applied to the middle surface of the shell; for thin shells, these terms are small and may be disregarded. The coefficients K_{4h} and K_{4c} refer to the case wherein the circumferential stress is induced by hydrostatic and constant directional pressure, respectively.

For any nodal pattern, the three positive roots Ω_1, Ω_2, Ω_3 of Eq. (7) are the frequency ratios of the first three modes of vibrations. The character of these modes is dependent upon the shell parameters and the mode shape. In the case of axisymmetric vibrations ($n = 0$), one of the three roots Ω_2 of Eq. (7) represents the frequency of the uncoupled torsional mode; the other two (nontorsional) modes involve predominantly longitudinal or radial motion, depending upon the geometry of the shell. Moreover, the lowest nontorsional mode Ω_1 may be greater or smaller than Ω_2 depending upon the R/L ratio. For modes having one or more circumferential waves, the lower mode of frequency $\Omega_1(\Omega_1 < \Omega_2 < \Omega_3)$ involves predominantly radial motion, whereas the other two involve predominantly longitudinal and tangential motion. It can be established by numerical evaluation that the effect of initial stresses on the frequency Ω_2 and Ω_3 is negligible. Taking this into account, it can be shown by writing (7) into a product form that the frequency of the first mode of a shell under the influence of initial stress is related to its three frequencies of free vibrations $\bar{\Omega}_1$, $\bar{\Omega}_2$, $\bar{\Omega}_3$ by the following relation:

$$\Omega_1^2 = \bar{\Omega}_1^2 - \frac{(K_4\bar{N} + K_5\bar{T})}{\bar{\Omega}_3^2\bar{\Omega}_2^2} \qquad (10)$$

This relation indicates that the change of $\bar{\Omega}_1^2$, due to the initial stresses considered, is proportional to the magnitude of these stresses. The frequencies of free vibrations for certain ranges of the shell parameters are obtainable from tables, such as those presented by Bleich and Baron.[11]

A considerable simplification is introduced if, as suggested by Reissner,[12] the effect of tangential and longitudinal inertia are disregarded. In this case, the frequency equation reduces to

$$\Omega_1^2 = -\frac{K_3 + K_4\bar{N} + K_5\bar{T}}{\bar{K}_2} \qquad (11)$$

where

$$\bar{K}_2 = \frac{2n^4h^4(1+\beta^2)^2}{(1-\nu)\pi^4R^4} \qquad (12)$$

For shells vibrating in modes having a large number of circumferential waves, $1/n^2$ can be disregarded as compared to unity, and the foregoing relation reduces to

$$\Omega_1^2 = \frac{2h^2}{(1-\nu)\pi^2R^2} \times \left[\frac{(1-\nu^2)\beta^4}{(1+\beta^2)^2} + \frac{s^2n^4}{3}(1+\beta^2)^2 + n^2\bar{N} + n^2\beta^2\bar{T}\right] \qquad (13)$$

This expression is identical to that obtained by Reissner[7] on the basis of the Marguerre shallow shell theory and may also be obtained from the Donnell equations[3] if only the radial inertia term is included.

Whenever the lowest root of Eq. (7) is of a much smaller order of magnitude than the others, on the basis of algebraic considerations, it may be approximated by

$$\Omega_1^2 = -\frac{K_3 + K_4\bar{N} + K_5\bar{T}}{K_2} \qquad (14)$$

Thin shells vibrating freely in modes whose ratio of circumferential to axial wavelength is small as compared to unity have one frequency of a much smaller order of magnitude than the other two. For these modes, the frequencies obtained on the basis of Eq. (14) are expected to constitute a satisfactory approximation. Moreover, β^2 and s^2n^2 can be disregarded as compared to unity resulting in the following expression for the case of hydrostatic pressure:

$$\Omega_1^2 = \frac{2n^2h^2}{(1-\nu)\pi^2R^2} \times \left[\frac{(1-\nu^2)\beta^4}{1+n^2} + \frac{s^2(n^2-1)^2}{3(n^2+1)} + \frac{n^2-1}{n^2+1}\bar{N} + \beta^2\bar{T}\right] \qquad (15)$$

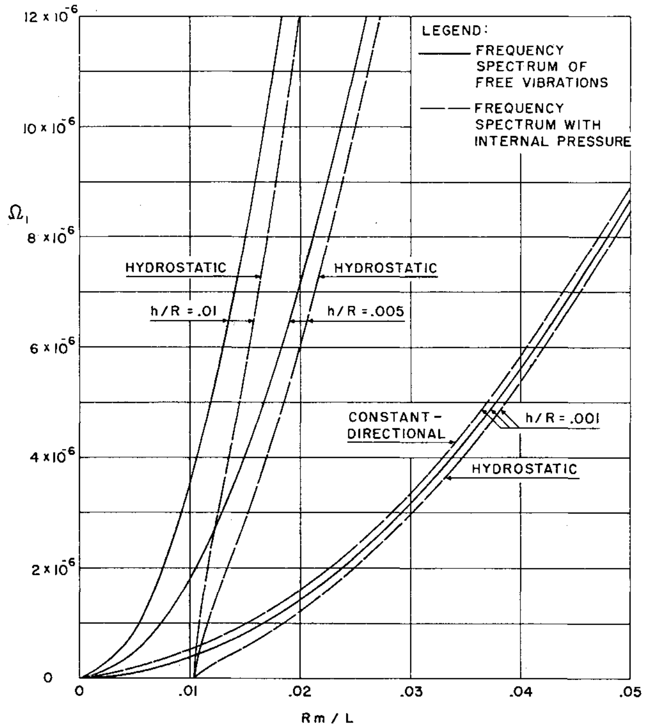

Fig. 1 Effect of internal pressure ($\bar{N}$ = 0.001) on the frequency of the beam-type flexural mode (n = 1).

In the case of constant directional pressure, we obtain

$$\Omega_1{}^2 = \frac{2n^2h^2}{(1-\nu)\pi^2R^2} \times \left[\frac{(1-\nu^2)\beta^4}{1+n^2} + \frac{s^2(n^2-1)^2}{3(n^2+1)} + \frac{(n^2-1)^2}{n^2(n^2+1)}\bar{N} + \beta^2\bar{T}\right] \quad (16)$$

The first term in the foregoing equation represents the membrane effect, the second, the effect of bending, and the last two, the effect of initial stresses. Notice that these expressions are not valid for $n = 1$ inasmuch as in this case, the effect of β^2 can not be disregarded.

Serbin,[13] by assuming that β^2 is negligible as compared to unity and by disregarding the effect on the strain energy of the $x\theta$ component of the shear strain and the circumferential component of the normal strain, obtained an expression for

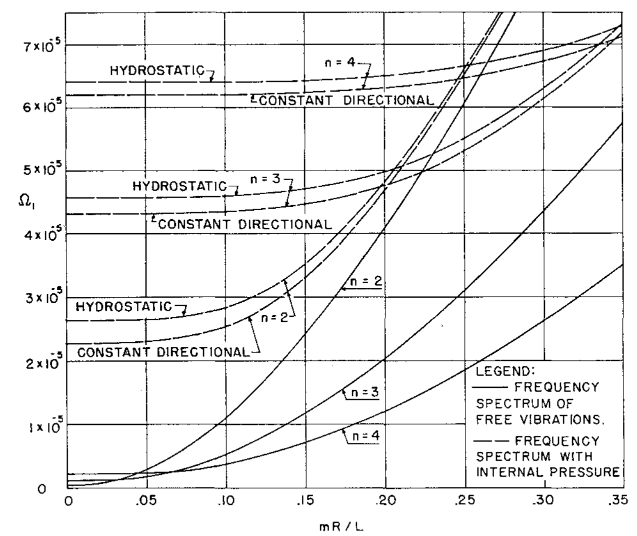

Fig. 2 Effect of internal pressure ($\bar{N}$ = 0.001) of the frequency Ω_1 of the lobar-type flexural modes of a shell with h/R = 0.001.

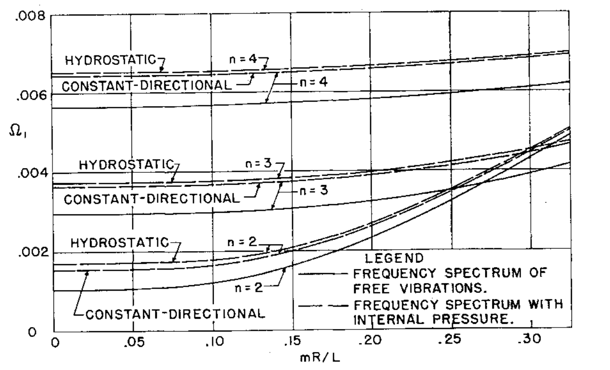

Fig. 3 Effect of internal pressure ($\bar{N}$ = 0.001) on the frequency Ω_1 of the lobar-type flexural modes for a shell with h/R = 0.05.

the frequency of the first mode of a shell under all around pressure $[T = (N/2)]$, which in our notation is

$$\Omega_1{}^2 = \frac{2n^2h^2}{(1-\nu)\pi^2R^2} \times \left[\frac{\beta^4}{1+n^2} + \frac{s^2(n^2-1)^2}{3(n^2+1)} + \frac{(n^2-1)^2}{n^2(n^2+1)}\bar{N}\right] \quad (17)$$

This formula, with the exception of the factor $(1 - \nu^2)$ of the first term, can be reduced from (16). It is apparent, therefore, that Serbin's formula yields the frequency of a shell under the influence of constant directional pressure and not as intended hydrostatic pressure. Therefore, it is not expected to yield accurate results for modes having a small number of circumferential waves and should be replaced by Eq. (15) presented in this investigation.

Numerical Evaluation and Comparison of Results

The frequency equation (7) obtained in this investigation and the one obtained by Reissner [Eq. (13)] are evaluated

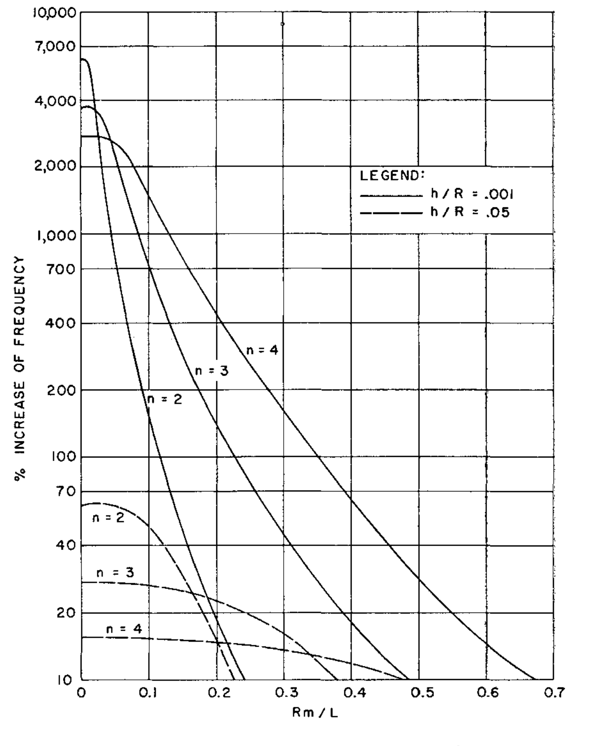

Fig. 4 Relative effect of internal hydrostatic pressure on the frequency Ω_1 of the lobar-type flexural modes.

for shells under the influence of only initial circumferential stress $\bar{N} = 0.001$ and only initial axial stress $\bar{T} = 0.001$. The Poisson's ratio is taken as 0.3. It is ascertained that the rather large initial stresses considered effect only the frequency Ω_1 of the first mode, which, with the exception of axisymmetric vibrations ($n = 0$), is the lower mode and involves predominantly radial motion. Furthermore, these initial stresses have a negligible effect on the mode shape.

Initial Circumferential Stress

The effect of initial circumferential stress due to internal pressure on the frequency of the beam-type flexural mode ($n = 1$) is illustrated in Fig. 1. It can be seen that internal hydrostatic pressure decreases the frequency of this mode; this effect is negligible for large values of (mR/L) [$(mR/L) > 0.5$]. However, it becomes very large for small values of mR/L, and the frequency can be decreased to zero, denoting that the shell reaches a condition of instability. This buckling phenomenon has been discussed by Flügge[5] who established the critical pressure as

$$p_{cr} = E\pi^2(Rh/L^2) \tag{18}$$

Note that the critical circumferential resultant stress $\bar{N}$ is independent of the h/R ratio. As can be seen in Fig. 1 or computed from Eq. (18), the critical mR/L ratio is, for $\bar{N} = 0.001$, approximately 0.0102.

Initial circumferential stress does not influence the frequency of the axisymmetric modes ($n = 0$). Its effect on the frequency of the lobar-type flexural modes ($n = 2, 3, 4, \ldots$) is illustrated in Figs. 2–4. It is apparent that initial internal hydrostatic pressure increases appreciably the frequency of these modes. This effect is larger for thinner shells and for modes having large axial wavelengths. For instance, the frequency Ω_1 of a shell made of steel with $h/R = 0.001$ and $mR/L = 0.03$ vibrating in a mode with $n = 2$ under the influence of 1-psi internal hydrostatic pressure is approximately 420 times the frequency of free vibrations $\bar{\Omega}_1$. This finding appears to be in contradiction with that of Fung et al.,[9] who indicated that initial stresses have a significant effect on the frequency of the first mode, only for $n > 3$. This discrepancy is principally due to the fact that the evaluation of the results in Ref. 9 was limited to rather large values of mR/L. As evident from Fig. 4, for example, for values of $Rm/L > 0.45$, the effect of initial stress is negligible for modes having less than four circumferential waves. For a specific value of the h/R and mR/L ratios, the relative effect of initial circumferential stress varies considerably with the number of circumferential waves; it is larger for modes having a number of circumferential waves close to the one into which a simply supported shell of length L/m will buckle. For instance, a shell with $h/R = 0.001$ will buckle under external pressure into three or four circumferential waves depending on whether $mR/L = 0.04$ or $mR/L = 0.05$, respectively (see Ref. 10). It is evident from Fig. 4 that the relative effect of initial circumferential stress in the first case is greatest for $n = 3$, whereas in the second case it is greatest for $n = 4$.

It is of interest to note the change of the frequency of the predominantly radial mode Ω_1 due to small alternations in the character of the applied pressure. For instance, it can be seen in Fig. 1 that the frequency of the beam-type flexural mode increases because of initial constant directional pressure, that is, a shell can not buckle under the influence of this pressure. In Figs. 2 and 3 it can be observed that initial constant directional pressure increases the frequency of the lobar-type flexural modes. This effect for modes having a small number of circumferential waves is smaller than that of hydrostatic pressure; the difference increases as the Rm/L and h/R ratios decrease.

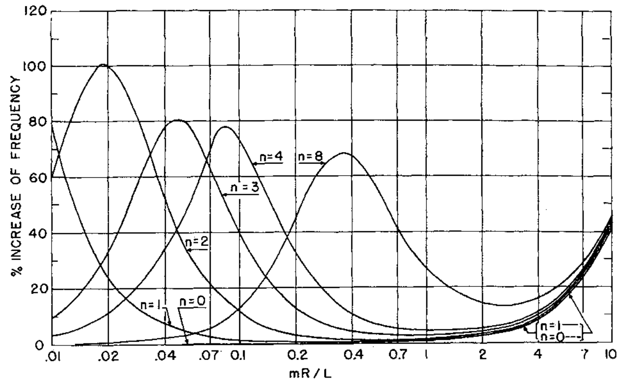

Fig. 6 Relative effect of initial axial tension ($\bar{T} = 0.001$) on the frequency Ω_1 of a shell with $h/R = 0.001$.

Initial Axial Tension

As illustrated in Figs. 5 and 6, initial axial tension increases the frequency Ω_1; however, its maximum effect is not as large as that of initial circumferential stress. For modes with $n \neq 1$, the influence of initial axial stress decreases as h/R increases. For example, for $h/R = 0.001$, this effect is appreciable, whereas for $h/R = 0.05$, it is less than 5%. For beam-type flexural vibrations ($n = 1$), the effect of initial axial stress on the frequency Ω_1 of modes having short axial wavelengths also decreases as h/R increases; however, for modes having long axial wavelengths, this effect is not dependent upon h/R. For relatively thick shells, therefore, only the frequency Ω_1 of the beam-type flexural vibrations of modes having long axial wavelengths is influenced significantly by initial axial stress. This behavior is compatible with the column-type mode of buckling of relatively thick shells under axial compression.

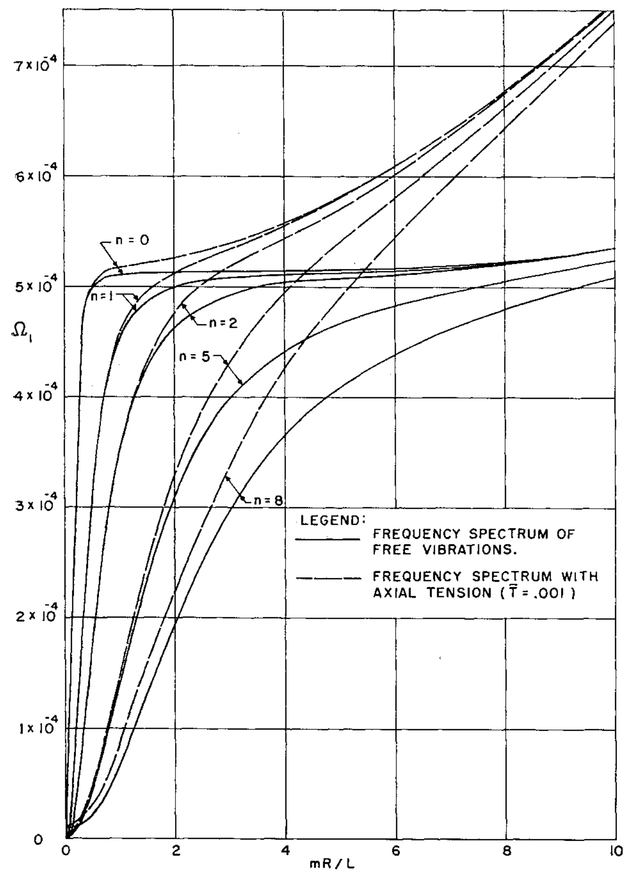

Fig. 5 Effect of initial axial tension ($T = 0.001$) on the frequency Ω_1 of a shell with $h/R = 0.001$.

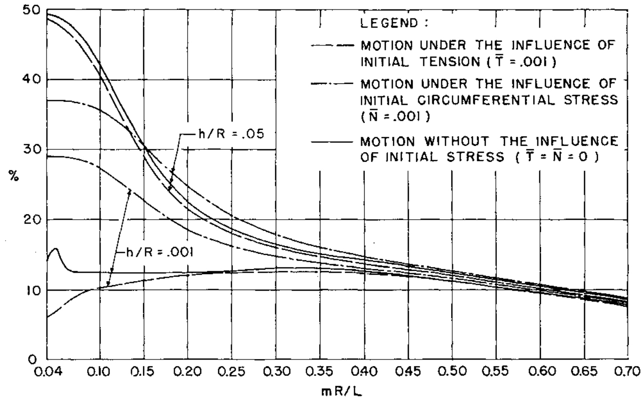

Fig. 7 Percentage by which Reissner's results exceed those presented in this investigation for $n = 2$.

For thin shells ($h/R = 0.001$) vibrating in modes having short axial wavelengths, the relative effect of initial axial stress on the frequency Ω_1 is not dependent upon the number of circumferential waves inasmuch as in these modes the effect of the axial wavelength on the frequency is of greater significance than that of the circumferential wavelength. In the other range of the frequency spectrum, as evident from Fig. 6, the relative effect of initial axial stress is negligible for axisymmetric modes but can become large for flexural modes. This relative effect reaches a maximum at a value of mR/L, which increases as the number of circumferential waves increases.

Comparison of Results

For thin shells vibrating in modes having short wavelengths, the results obtained on the basis of Eq. (7) are in complete accord with those obtained on the basis of the Reissner frequency equation (13). As previously noted, for these modes the effect of the axial wavelengths on the frequency is of greater significance than that of the circumferential wavelength, and, consequently, shallow shell theories yield accurate results even for modes having a small number of circumferential waves ($n = 0, 1, 2, 3$). Furthermore, the motion associated with these modes is predominantly flexural, and, therefore, the axial and tangential inertia neglected by Reissner has an insignificant effect on the frequency Ω_1. However, it should be emphasized that, for vibrations involving very short wavelengths, the influence of higher modes (not included in the theory employed in this investigation) on the frequency becomes significant, and, consequently, the results are not valid. In this range, theories including the effect of shear deformation and rotatory inertia yield more satisfactory results.

Axisymmetric and beam-type modes ($n = 0, 1$) having rather long axial wavelengths involve considerable tangential (in case $n = 1$) and axial (in case $n = 0$) displacements; therefore, the effect of longitudinal and tangential inertia cannot be disregarded. Moreover, for these modes shallow shell theories do not yield accurate results, and, concommitantly, Reissner's frequency equation is not valid. For the lobar-type flexural modes, the divergence of the results decreases as the number of circumferential waves increases and becomes negligible for large values of n. Furthermore, as illustrated in Fig. 7, for $n = 2$, this divergence of the results becomes more pronounced for thicker shells and for modes having long axial wavelengths.

Therefore, it may be concluded that the frequency equation presented by Reissner yields acceptable results not only for modes having a large number of circumferential waves (as originally intended) but also for modes having a small number of circumferential waves, provided that their axial wavelengths are rather short. Reissner's equation is not valid for shells vibrating in modes having a small number of circumferential waves together with rather long axial wavelengths. This case is of considerable practical interest, however, for, as shown in Figs. 2 and 3, in this range of shell parameters, the fundamental frequency may correspond to modes having a small number of circumferential waves.

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